Econ 802
Final Exam
Answer Key

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1\. (a) Let (y, x) be optimal at (p, w) so $\pi(p,w) = py - wx$.
Let (y', x') be optimal at (p', w) so $\pi(p', w) = p'y' - wx'$.
Let (y'', x'') be optimal at (p'', w) so $\pi(p'', w) = p''y'' - wx''$
with $p'' = tp + (1-t)p'$ where $0 \le t \le 1$.

We need to show that $\pi(p'', w) \le t\pi(p,w) + (1-t)\pi(p', w)$
or $[tp + (1-t)p']y'' - wx'' \le t[py - wx] + (1-t)[p'y' - wx']$
Write $wx'' = twx'' + (1-t)wx''$. Then we want

$$0 \le t\underbrace{\left[(py - wx) - (py'' - wx'')\right]}_{\text{This is } \ge 0 \text{ because } (y,x) \text{ is optimal at } (p,w)} + (1-t)\underbrace{\left[(p'y' - wx') - (p'y'' - wx'')\right]}_{\text{This is } \ge 0 \text{ because } (y',x') \text{ is optimal at } (p',w)}$$

So we have proven the convexity.

(b) If x^* maxes profit then it satisfies the FOC
$p \frac{\partial f(x^*)}{\partial x_i} = w_i$ for all i.

If profit is positive then $p f(x^*) - \sum_i w_i x_i^* > 0$.

Returns to scale are locally decreasing when

$$\frac{\sum_i \frac{\partial f(x^*)}{\partial x_i} x_i^*}{f(x^*)} < 1 \Rightarrow \underbrace{\sum_i w_i x_i^*}_{\text{substitute from FOC.}} < p f(x^*) \text{ which is true.}$$

(c) Write the cost function as

$$c(w,y) = \sum_i w_i x_i(y) - \lambda(y)\left[f[x_1(y)\ldots x_n(y)] - y\right]$$

where $x_i(y)$ is the conditional input demand at output y and $f[x(y)] \equiv y$ from the FOC.

Differentiating with respect to y gives

$$\frac{\partial c(w,y)}{\partial y} = \sum_i w_i \frac{\partial x_i(y)}{\partial y} - \lambda(y)\left[\sum_i \frac{\partial f[x(y)]}{\partial x_i}\frac{\partial x_i}{\partial y}\right]$$
$$+ \lambda(y) - \frac{\partial \lambda(y)}{\partial y}\left[f(x(y)) - y\right].$$

From FOC,

$$\underbrace{\left[w_i - \lambda(y)\frac{\partial f[x(y)]}{\partial x_i}\right]}_{=0}\frac{\partial x_i(y)}{\partial y} = 0 \text{ for all } i$$

and $f[x(y)] = y$ so

$$\frac{\partial c(w,y)}{\partial y} = \lambda(y) \text{ and } MC = \text{Lagrange multiplier.}$$

2. (a) Suppose $f(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$ which is a CES function.

For cost min, set up the Lagrangean

$$L = w_1 x_1 + w_2 x_2 - \lambda\left[(x_1^\rho + x_2^\rho)^{1/\rho} - y\right]$$

FOC: $$\frac{\partial L}{\partial x_1} = w_1 - \lambda\left(\frac{1}{\rho}\right)(x_1^\rho + x_2^\rho)^{1/\rho - 1}\rho x_1^{\rho-1} = 0$$

$$\frac{\partial L}{\partial x_2} = w_2 - \lambda\left(\frac{1}{\rho}\right)(x_1^\rho + x_2^\rho)^{1/\rho - 1}\rho x_2^{\rho-1} = 0$$

Divide the first equation by the second to get

$$\frac{w_1}{w_2} = \left(\frac{x_1}{x_2}\right)^{\rho-1} \quad \text{or} \quad \frac{x_1}{x_2} = \left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}}$$

(Note that this only makes sense if $\rho < 1$ because $\frac{x_1}{x_2}$ must be a <u>decreasing</u> function of $\frac{w_1}{w_2}$.)

The elasticity of substitution is $\sigma = -\frac{d\left(\frac{x_1}{x_2}\right)}{d\left(\frac{w_1}{w_2}\right)} \cdot \frac{\left(\frac{w_1}{w_2}\right)}{\left(\frac{x_1}{x_2}\right)}$

$$= \frac{-\left(\frac{1}{\rho-1}\right)\left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}-1}\left(\frac{w_1}{w_2}\right)}{\left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}}} = \frac{1}{1-\rho} > 0$$

which is a constant.

(b) If the production function $f(x)$ is homothetic then we can write $f(x) = g[h(x)]$ where $g' > 0$ and $h(x)$ is homogeneous of degree one.

For cost min we use

$$L = w_1 x_1 + w_2 x_2 - \lambda\left[g[h(x_1, x_2)] - y\right]$$

FOC:
$$\left.\begin{aligned} w_1 - \lambda g' \frac{\partial h(x^*)}{\partial x_1} &= 0 \\ w_2 = \lambda g' \frac{\partial h(x^*)}{\partial x_2} &= 0 \end{aligned}\right\} \Rightarrow \frac{w_1}{w_2} = \frac{\frac{\partial h(x_1^*, x_2^*)}{\partial x_1}}{\frac{\partial h(x_1^*, x_2^*)}{\partial x_2}}$$

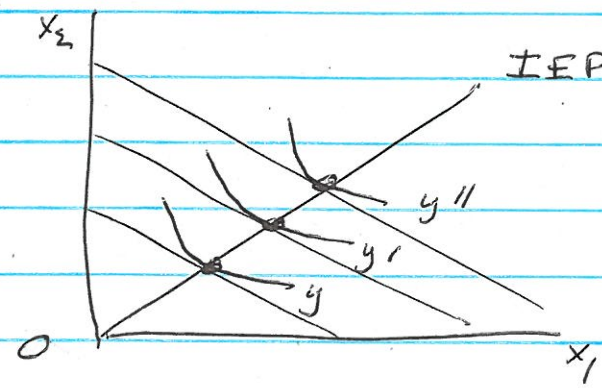
Since h is homogeneous of degree one, its derivatives are homogeneous of degree zero. Therefore we can write
$$\frac{\partial h(x_1^*, x_2^*)}{\partial x_1} = \frac{\partial h\left(\frac{x_1^*}{x_2^*}, 1\right)}{\partial x_1}$$

and
$$\frac{\partial h(x_1^*, x_2^*)}{\partial x_2} = \frac{\partial h\left(\frac{x_1^*}{x_2^*}, 1\right)}{\partial x_2}$$

Since each derivative depends only on the ratio $\frac{x_1^*}{x_2^*}$
the same is true for the ratio of the derivatives.
Thus the equation

$$\frac{w_1}{w_2} = \frac{\dfrac{\partial h(\frac{x_1^*}{x_2^*}, 1)}{\partial x_1}}{\dfrac{\partial h(\frac{x_1^*}{x_2^*}, 1)}{\partial x_2}}$$

determines the ratio $\frac{x_1^*}{x_2^*}$ regardless of the output level y.
Because the input ratio is the same for all values of y,
the input expansion path is linear:

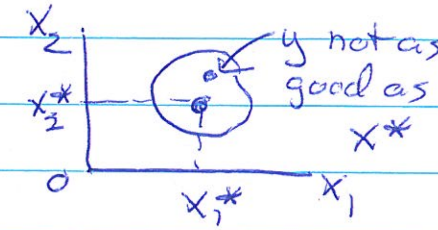


where the slope of IEP is $\frac{x_2^*}{x_1^*}$.

(c) Let $y = \min\{ax_1, bx_2\}$ with $a > 0$, $b > 0$.
Profit max requires cost min, so first let's solve for the cost function. We have $ax_1 = bx_2 = y$ so
$c(w, y) = y\left[\frac{w_1}{a} + \frac{w_2}{b}\right]$
Then profit as a function of output is
$py - c(w, y) = y\left[p - \frac{w_1}{a} - \frac{w_2}{b}\right]$

If $p - \frac{w_1}{a} - \frac{w_2}{b} > 0$ then profit is unbounded because we can make y as large as we want $\Rightarrow$ no solution.
If $p - \frac{w_1}{a} - \frac{w_2}{b} = 0$ then profit is zero for all $y \geq 0$.
If $p - \frac{w_1}{a} - \frac{w_2}{b} < 0$ then profit is maximized at $y = 0$ and again profit is zero (it would be negative for any $y > 0$).

3 (a) Local non-satiation – No. Consider the point x^*. For every $\epsilon > 0$, the points $y \neq x^*$ in the set such that $|y - x^*| < \epsilon$ all have $x^* \succ y$.

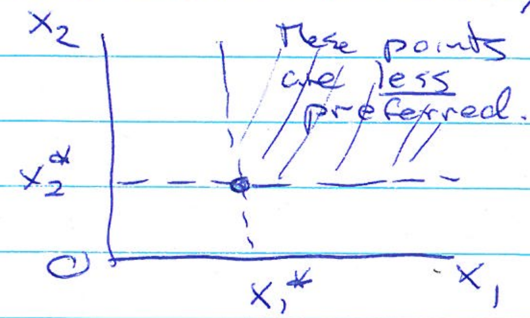


Weak monotonicity – No

This requires that if $x \geq y$ then $x \succsim y$. This is clearly untrue if $y = x^*$ because all points to the northeast of x^* are less preferred.

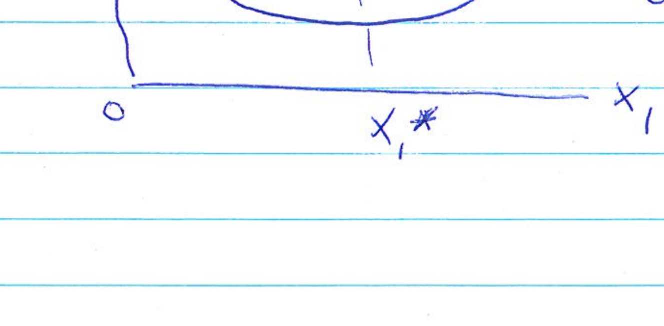
Strong monotonicity – No

This requires that if $x \geq y$ and $x \neq y$ then $x \succ y$. This is untrue for the same reason as above for weak monotonicity



Strict Convexity – Yes

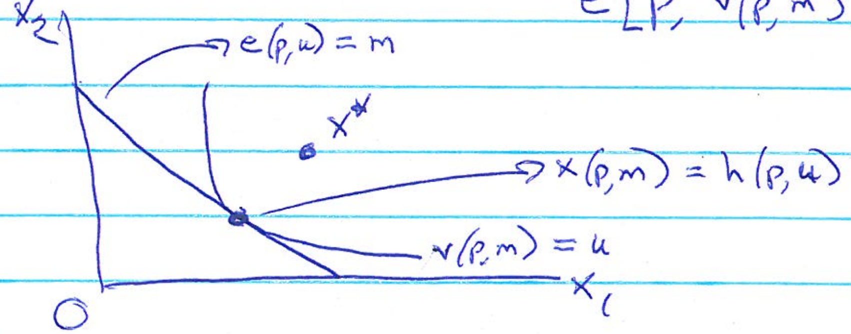
George's indifference curves are circles. If you choose any two points on the boundary or in the interior of a circle, the line segment between the points will be in the interior of the circle.



If $x \succsim z$ and $y \succsim z$ then all points on the line segment $tx + (1-t)y$ with $0 < t < 1$ are strictly preferred to z.

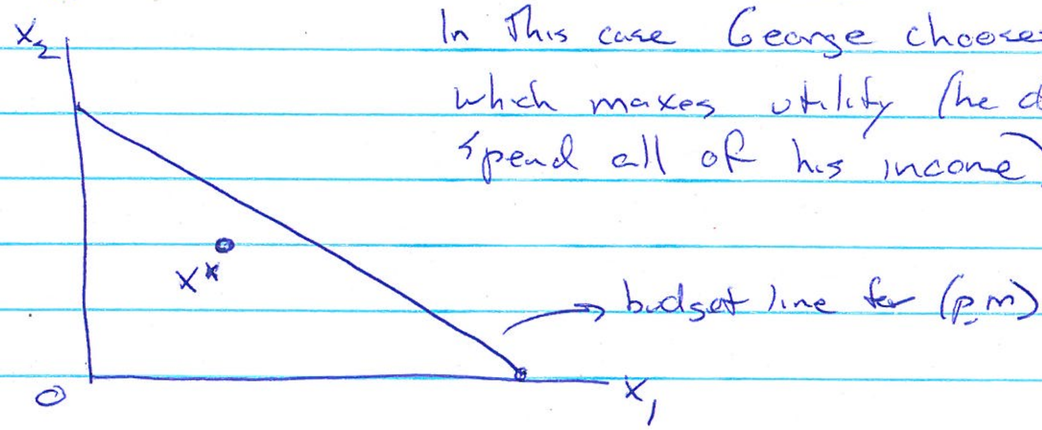
3 (b) There are two cases to consider. If the budget line passes below or through x^*, it is easy to see that the usual identities hold:

$$x(p,m) \equiv h[p, v(p,m)]$$
$$h(p,u) \equiv x[p, e(p,u)]$$
$$v[p, e(p,u)] \equiv u$$
$$e[p, v(p,m)] \equiv m$$



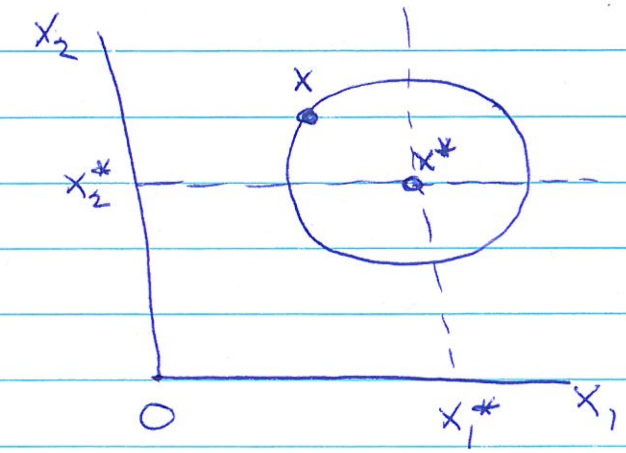
Now suppose the budget line passes <u>above</u> x^*:

In this case George chooses x^* which maxes utility (he does not spend all of his income)



The Marshallian demand is $x(p,m) = x^*$. The resulting utility level is $v(p,m) = u(x^*) = 0$. The Hicksian demand at a utility $u=0$ is x^* because no other point is feasible in the expenditure min problem. So $x(p,m) = x^* = h[p, v(p,m)]$. However $v(p,m) = 0$ gives $e[p, v(p,m)] = e(p,0) < m$ because to achieve zero utility George doesn't need the income m; a smaller income such that the budget line passes through x^* would achieve this. So the identity $e[p, v(p,m)] \equiv m$ does <u>not</u> always hold.

3(c) No point (x_1, x_2) that has $x_1 \geq x_1^*$ or $x_2 \geq x_2^*$ or both can be an optimal bundle for any prices and income $(p, m) > 0$. Consider the following graph:

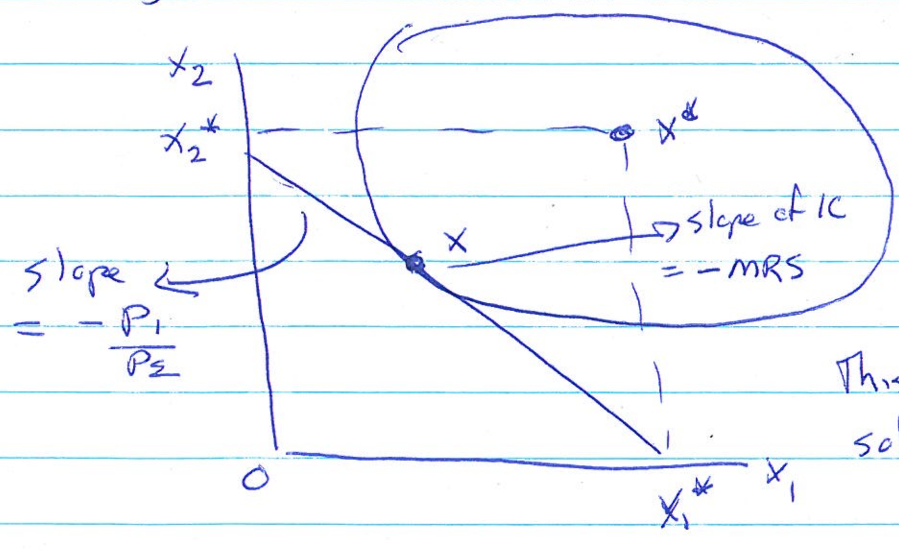


At a point like x, the indiff curve has a positive slope. Any budget line with a negative slope that passes through this point will have some points that are closer to x^* than x, and therefore preferred to x, so x will not be chosen.

The same is true for points with $x_1 \geq x_1^*$ and $x_2 \leq x_2^*$. No point with $x > x^*$ can be optimal because if it is feasible, George can choose x^* instead, which he prefers.

[Note: x^* is optimal for any budget line passing through x^*, but $p(x^*)$ is not uniquely defined]

So the only points for which $p(x)$ is well-defined are those with $x_1 < x_1^*$ and $x_2 < x_2^*$. In this region George's indifference curves slope down and are strictly convex, so for any given x we can construct a budget line passing through x that is tangent to the indifference curve at x. Graphically:



MRS at x is $\frac{MU_1}{MU_2} = \frac{x_1^* - x_1}{x_2^* - x_2}$

So set $\frac{p_1}{p_2} = \frac{x_1^* - x_1}{x_2^* - x_2}$

with $p_1 x_1 + p_2 x_2 = m = 1$.

This pair of equations can be solved for $p_1(x)$ and $p_2(x)$.

4. (a) max $\sum_i b_i(x_i) + \sum_i y_i$

subject to $\sum_i x_i = \sum_j q_j$ and $\sum_i y_i = \sum_i m_i - \sum_j c_j(q_j)$

$$L = \sum_i b_i(x_i) + \sum_i m_i - \sum_j c_j(q_j) - \lambda\left[\sum_i x_i - \sum_j q_j\right]$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x_i} &= b_i'(x_i) - \lambda = 0 \quad \text{all } i = 1 \ldots n \\ \frac{\partial L}{\partial q_j} &= -c_j'(q_j) + \lambda = 0 \quad \text{all } j = 1 \ldots m \end{aligned}\right\} \text{FOC}$$

Let $x_i = d_i(\lambda)$ for $i = 1 \ldots n$ be i's "demand" for x_i

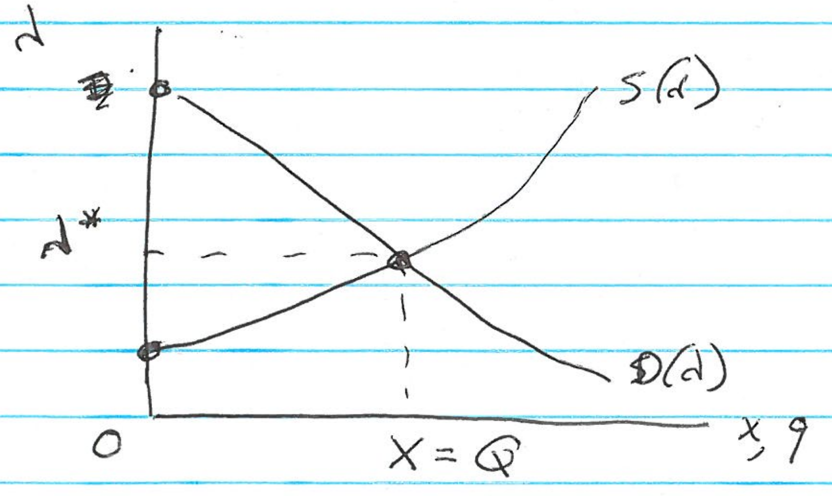
Let $q_j = s_j(\lambda)$ for $j = 1 \ldots m$ be j's "supply" for q_j

(These are obtained implicitly from FOC)

To satisfy the feasibility constraint $\sum_i x_i = \sum_j q_j$

we must have $\sum_i d_i(\lambda) = \sum_j s_j(\lambda)$

or simply $D(\lambda) = S(\lambda)$ where aggregate demand = aggregate supply where λ plays the role of a price.



At λ^* we have "demand" = "supply" and the aggregate feasibility constraint is satisfied. This gives the solution for the multiplier.

4(b) Consumer i's budget constraint is

$$p x_i + y_i = m_i + \sum_j T_{ij} \pi_j$$

The aggregate excess demands are

$$z_x(p) = \sum_i x_i(p) - \sum_j q_j(p)$$

$$z_y(p) = \sum_i y_i(p) - \sum_i m_i + \sum_j c_j(q_j(p))$$

Note: This is demand for the y good from firms.

$$\Rightarrow \quad p z_x(p) + z_y(p) = \sum_i p x_i(p) - \sum_j p q_j(p) + \sum_i y_i(p) - \sum_i m_i + \sum_j c_j(q_j(p))$$

From the budget constraints, $\sum_i p x_i(p) + \sum_i y_i(p) = \sum_i m_i + \sum_j \pi_j(p)$

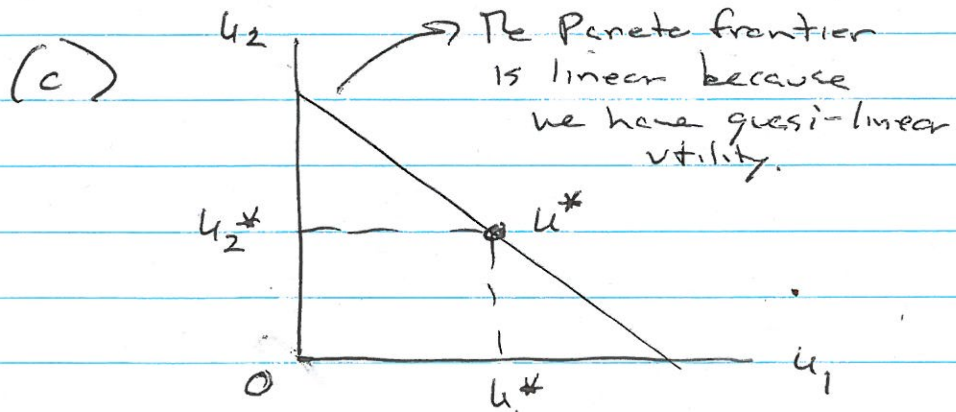
(where we use $\sum_i T_{ij} = 1$ for all j)

Substituting, we get

$$p z_x(p) + z_y(p) = \sum_j \pi_j(p) - \Big[\underbrace{\sum_j (p q_j(p) - c_j(q_j(p)))}_{\text{because this is the sum of profits.}} \Big] = 0$$

Interpretation: This is Walras's Law.

(c)



Pick some arbitrary Pareto efficient pair of utilities (u_1^*, u_2^*). The Second welfare Theorem says we should be able to construct a Walrasian equilibrium that gives this result.

We know we have to maximize the sum of the utilities to get to the frontier, so we have FOC as in part (a). This and the solution for λ^* determine x_i^* for $i=1,2$ and all of the firm outputs q_j^*, $j=1 \ldots m$. We use the multiplier $p^* = \lambda^*$ as the equilibrium price of the x good, with the price of the y good $=1$ (numeraire).

Now we need $u_1^* = b_1(x_1^*) + y_1^*$

$$u_2^* = b_2(x_2^*) + y_2^*.$$

We can accomplish this by setting $y_1^* = u_1^* - b_1(x_1^*)$ and then constructing the budget $y_2^* = u_2^* - b_2(x_2^*)$ constraints so that

$$p^* x_i^* + y_i^* = m_i + \sum_j T_{ij} \pi_j^* \quad \text{where } \underbrace{\pi_j^* = p^* q_j^* - c_j(q_j^*)}$$

For example, we could set $T_{ij} = 1/2$ for all i, j and then assign the endowments of the y good so that $m_i = p^* x_i^* + y_i^* - \frac{1}{2} \sum_j \pi_j^*$.

Note that profit is max at q_j^* because $p^* = c_j'(q_j^*)$ from part (a).

It can be shown that if we sum over i, these endowments are feasible.

5 (a) Consumer i solves $\max \; a_i \ln r_i + b_i \ln L_i$

subject to $p r_i = w(1 - L_i) + \sum_j \left(\frac{1}{n}\right) \pi_j$

or $p r_i + w L_i = \underbrace{w + \sum_j \left(\frac{1}{n}\right) \pi_j}_{\text{Define this to be } m_i}$

$$\text{FOC:} \quad \left.\begin{array}{l} \dfrac{a_i}{r_i} - \lambda_i p = 0 \\[2ex] \dfrac{b_i}{L_i} - \lambda_i w = 0 \end{array}\right\} \Rightarrow \begin{array}{l} r_i(p, m_i) = \dfrac{a_i m_i}{p} \Big/ (a_i + b_i) \\[2ex] L_i(w, m_i) = \dfrac{b_i m_i}{w} \Big/ (a_i + b_i) \end{array}$$

To get the Marshallian demands as functions of (p, w) replace m_i using the budget constraint:

$$r_i(p,w) = \frac{a_i}{p}\left[w + \sum_j \left(\tfrac{1}{n}\right)\pi_j(p,w)\right] \Big/ (a_i + b_i)$$
$$L_i(p,w) = \frac{b_i}{w}\left[w + \sum_j \left(\tfrac{1}{n}\right)\pi_j(p,w)\right] \Big/ (a_i + b_i)$$

(b) To have WE, the firms must be maximizing profit. For firm j, profit $= py_j - wz_j = z_j[pk - w]$

If $pk > w$, there is no solution to the profit max problem.

If $pk < w$, every firm produces zero, which will not give an equilibrium (demand for rhubarb will be positive and there are no endowments of rhubarb).

So we must have $pk = w$ or $k = \frac{w}{p}$. This is the equilibrium price ratio (note that by homogeneity we cannot solve for the levels of p and w).

Now use the Marshallian demands with $\pi_j = 0$ for all j to get $R(p,w) = \sum_i r_i(p,w) = k \sum_i \frac{a_i}{(a_i+b_i)}$ = total demand for rhubarb

$L(p,w) = \sum_i L_i(p,w) = \sum_i \frac{b_i}{(a_i+b_i)}$ = total demand for leisure

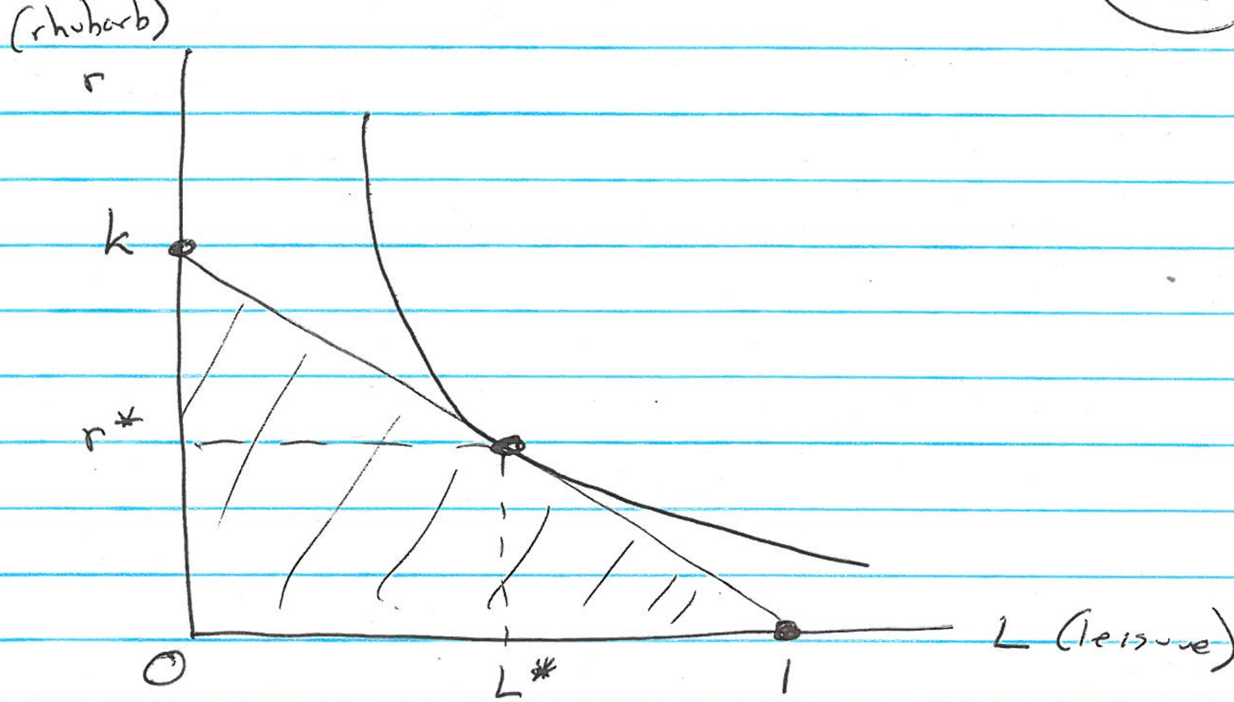
~~Note: for this to make sense we need $b_i \le 1$ for all i so leisure does not exceed time endowment.~~

$Z(p,w) = \sum_j z_j(p,w)$ where $\sum_j z_j = \sum_j \left(\frac{y_j}{k}\right)$

and $\sum_j y_j = R(p,w) = k \sum_i \frac{a_i}{(a_i+b_i)}$ (This clears the rhubarb market and firms are indifferent about their outputs due to CRS)

so $Z(p,w) = \sum_i a_i / (a_i + b_i)$ is total demand for labor.

5 (c)



The budget constraint is $pr + wL = w$ (since profit is zero)
where $k = \frac{w}{p} \Rightarrow r + kL = k$
So the consumer can have $r = 0$ and $L = 1$
or $r = k$ and $L = 0$

The budget constraint is linear as usual. Subject to this constraint, maximum utility occurs at (L^*, r^*) so these are the demands for rhubarb and leisure.
The isoprofit lines have $\pi = pr - wz \Rightarrow r = \frac{\pi}{p} + \frac{w}{p} z$ but in equilibrium this reduces to $r = kz$ where $\pi = 0$ and $z =$ labor. The firm's production possibilities set is the shaded area in the graph. It has a linear boundary and the highest isoprofit line coincides with this boundary. So in equilibrium the firm is indifferent among the points along this line (they all give zero profit) but it is happy to produce output r^* using the labor input $z^* = \frac{r^*}{k}$ and this clears both the rhubarb market and the labor market.